MINIMAL UNBIASED DESIGNS FOR LINEAR PARAMETRIC FUNCTIONS

by

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1. Introduction

In this paper the concept of minimal unbiased designs for estimating linear parametric functions is introduced. Characterizations of these designs are given for the case where no assumption is made on the total parametric vector and for the case where some elements of this vector are assumed to be zero. These minimal unbiased designs then lead to a class of unbiased designs for estimating linear parametric functions. Examples are given to illustrate the concepts and the developments. Prior to presenting the results some introductory definitions, notations, and concepts are presented.

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2. Preliminary Definitions and Notations

For the t factors under the control of the experimenter, let there be k_i levels for the i^{th} factor. The total number of treatments (combinations) in the full factorial is equal to $N = \prod_{i=1}^{t} k_i$. Let T denote the set of all treatments in the experiment.

Definition 2.1: A factorial arrangement Γ with parameters $k_1, k_2, \ldots, k_t$; m; n; $r_1, r_2, \ldots, r_N$ is defined to be a collection of treatments of T such that the j^{th} treatment in T has multiplicity $r_j \geq 0$, with at least one nonzero r_j; m is the number of nonzero r_j's, and $n = \sum_{j=1}^{N} r_j$. We denote such a factorial arrangement by the symbol FA($k_1, k_2, \ldots, k_t$; m; n; $r_1, r_2, \ldots, r_N$). Note that in statistical design terminology the multiplicity r_j is referred to as the replication number of the j^{th} treatment.

Definition 2.2: A factorial arrangement is said to be complete if $r_j > 0$ for all j.

Definition 2.3: A complete factorial arrangement is said to be minimal if $r_j = 1$ for all j and is designated by MFA($k_1, k_2, \ldots, k_t$) or simply MFA if there is no ambiguity.

Definition 2.4: A factorial arrangement is said to be a fractional factorial arrangement, or more simply a fractional replicate, if some but not all $r_j > 0$. We denote a fractional replicate by FFA($k_1, k_2, \ldots, k_t$; m; n; $r_1, r_2, \ldots, r_N$), or more simply as FFA(m; n; $r_1, r_2, \ldots, r_N$) whenever the underlying factorial arrangement is clear.

With each treatment g in T we associate a random variable y_g, which is called an observation, a response, or a measurement, with $E[y_g] = \theta' f(g)$, where θ is a vector of k unknown parameters called factorial effects; f is a vector of k continuous real-valued known functions on the collection of g's in T. In matrix notation the linear model $E[y_g] = \theta' f(g)$ can be written as:

$$(2.1) \qquad E\left[Y_T\right] = W_T\theta, \quad \text{Cov } Y_T = \sigma^2 V_T,$$

where V_T is a known positive definite $k \times k$ matrix which with no loss in generality will be assumed to be the identity matrix of order k. The element in the g^{th} row and j^{th} column of W_T is equal to $f_j(g)$. The $N \times k$ matrix W_T is known as the design matrix in the literature. This type of model is often used in practice with a celebrated one being the polynomial model. Note that (2.1) is a linear model associated with an MFA.

Corresponding to a factorial arrangement Γ, there is an observational system induced by (2.1), namely,

$$(2.2) \qquad E\left[Y_\Gamma\right] = X_\Gamma\theta$$

where Y_Γ is the n-vector of observations associated with the m treatments in Γ. The $n \times k$ matrix X_Γ is simply read off from W_Γ taking repetitions of treatments in Γ into account.

Let ρ be the minimal complete factorial and Y_ρ be the corresponding $N \times 1$ observation vector. The observation vector may be written as

$$(2.3) \qquad Y_\rho = X_\rho\beta_\rho + \epsilon_\rho,$$

where $E[\epsilon_\rho] = 0$, $\text{Cov}(\epsilon_\rho) = \sigma^2 I$, the observations in Y_ρ are lexicographically ordered, and the coefficients in X_ρ are those for the polynomial model such that $X_\rho' X_\rho = I$.

3. Least Squares Estimation of β_ρ and of Linear Parametric Functions

Applying the least square procedure to equation (2.3), we obtain the following estimator for β_ρ:

$$(3.1) \qquad \hat{\beta}_\rho = \left(X_\rho' X_\rho\right)^{-1} X_\rho' Y_\rho = X_\rho' Y_\rho$$

with covariance $\sigma^2 I$.

Let Γ be the factorial arrangement $FA(k_1, k_2, \ldots, k_t;\ m;\ n;\ r_1, r_2, \ldots, r_N)$ associated with a polynomial model as follows:

$$(3.2) \qquad E\left[Y_\Gamma\right] = X_\Gamma \beta_\rho$$

where the elements of X_Γ contain the r_j replications for the j^{th} treatment. The $n \times N$ matrix X_Γ is obtained from the matrix X_ρ by taking repetitions into account.

Suppose that the experimenter is interested in estimating a set of linear parametric functions specified by $L\beta_\rho$ where L is a matrix of order $s \times N$ of rank $s \le N$. We shall distinguish between the following two cases which are treated successively:

Case 1. No specific *a priori* assumptions on the components of β_ρ. Let Γ be such that $L\beta_\rho$ is estimable, i.e., there exists a matrix K such that

$$(3.3) \qquad L = K_\Gamma X_\Gamma \quad .$$

The least squares estimator for $L\beta_\rho$ is

$$\widehat{L\beta}_\rho = L\left(X'_\Gamma X_\Gamma\right)^- X'_\Gamma Y_\Gamma \tag{3.4}$$

where $(A)^-$ denotes the generalized inverse of A. The covariance of $\widehat{L\beta}_\rho$ is

$$\mathrm{Cov}\left(\widehat{L\beta}_\rho\right) = L\left(X'_\Gamma X_\Gamma\right)^- L'\sigma^2 = K_\Gamma X_\Gamma\left(X'_\Gamma X_\Gamma\right)^- X'_\Gamma K'_\Gamma\sigma^2 \ . \tag{3.5}$$

The matrix $X_\Gamma(X'_\Gamma X_\Gamma)^- X'_\Gamma$ is invariant under any choice of a generalized inverse for $X'_\Gamma X_\Gamma$.

This case includes response surface estimation and prediction by setting $L = X_\Gamma$. Since $E[\widehat{L\beta}_\rho] = E[\widehat{X_\Gamma\beta_\rho}] = X_\Gamma\beta_\rho = E[Y_\Gamma]$, the estimator $\widehat{L\beta}_\rho$ is written as $\widehat{E[Y_\Gamma]}$.

<u>Case 2</u>. The experimenter has <u>a priori</u> knowledge of the exact values of some components of β_ρ. We may assume without loss of generality that these values are zero and $\beta'_\rho = [\beta'_1 \,\vdots\, \beta'_2] = [\beta'_1 \,\vdots\, 0]$. For this case (3.2) reduces to:

$$E\left[Y_\Gamma\right] = \left[X_{\Gamma_1} \,\vdots\, X_{\Gamma_2}\right]\begin{bmatrix}\beta_1\\ \cdots\\ 0\end{bmatrix} = X_{\Gamma_1}\beta_1 \ . \tag{3.6}$$

Let L be such that $L_1\beta_1$ is estimable, i.e., there exists a K_{Γ_1} such that

$$L_1 = K_{\Gamma_1}X_{\Gamma_1} \tag{3.7}$$

The least squares estimator of $L_1\beta_1$ together with its covariance matrix is:

(3.8) $$L_1\hat{\beta}_1 = L_1\left(X'_{\Gamma_1}X_{\Gamma_1}\right)^{-}X'_{\Gamma_1}Y_{\Gamma}$$

and

(3.9) $$\mathrm{Cov}\left(\widehat{L_1\beta_1}\right) = L_1\left(X'_{\Gamma_1}X_{\Gamma_1}\right)^{-}L_1'\sigma^2 .$$

Note that $L_1\hat{\beta}_1$ is unbiased as was $L\hat{\beta}_\rho$ previously. However, if $\beta_2 \neq 0$ then $L_1\hat{\beta}_1$ is no longer unbiased.

In selecting a design it is realistic to impose the condition that the design is capable of providing an unbiased estimator of $L\beta_\rho$, i.e., $L\beta_\rho$ is estimable.

Definition 3.1: If a design is such that $L\beta_\rho$ is estimable, it is denoted as an unbiased design. The class of all such designs is denoted by $\Delta(L)$ and will be referred to as the class of unbiased designs.

4. The Problem Of Characterizing Unbiased Designs

A preliminary problem in the study and use of factorial arrangements should be the characterization of the class of all unbiased designs $\Delta(L)$, with respect to the given $L\beta_\rho$. Let Γ be a design in $\Delta(L)$ and let X_Γ be the design matrix associated with Γ. The available theory in linear estimation states that $L\beta_\rho$ is estimable if and only if L is in the row space of X_Γ. Clearly, this tells us little of "immediate use" about which treatments should be in $\Delta(L)$. What researchers on linear models do is the following: they pick a design such that β_ρ is estimable which in turn guarantees estimability of $L\beta_\rho$. This means that Γ be at least a minimal complete factorial arrangement. Of course, all of these designs are contained in $\Delta(L)$, but they do not exhaust $\Delta(L)$, if L is not the identity matrix. For example, if L is a $1 \times N$ matrix then $\Delta(L)$ can contain designs of any number of distinct treatments from 1 to N inclusive. The lower bound is clearly achieved whenever $L_{1 \times N}$ is a multiple of a row of X_ρ for the minimal complete factorial arrangement ρ.

Consider now a general $L_{s \times N}$. A design containing treatments corresponding to rows of X_ρ having nonzero coefficients in the linear combinations clearly is unbiased. In other words if ℓ_i' is the i^{th} row of L of the form

$$\ell_i' = \sum_{j=1}^{N} \alpha_{ij}(R_j(\rho))$$

where $R_j(\rho)$ is the j^{th} row of X_ρ and if Γ_i is a design consisting of those treatments corresponding to the $R_j(\rho)$'s in ℓ_i' having nonzero α_{ij}'s, then the design containing the union of the Γ_i's is an unbiased design.

The following example illustrates the above concepts.

<u>Example 4.1.</u> Consider the 2 x 2 MFA with the model (3.2):

$$E\begin{bmatrix} y_{00} \\ y_{10} \\ y_{01} \\ y_{11} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \phi_1^0\phi_2^0 \\ \phi_1^1\phi_2^0 \\ \phi_1^0\phi_2^1 \\ \phi_1^1\phi_2^1 \end{bmatrix} = X_\rho \beta_\rho \quad .$$

Let $L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ and suppose the experimenter is interested in estimating $L\beta_\rho$. The traditional linear model theory says that one needs an arrangement containing the minimal complete factorial arrangement ρ, i.e., a design containing all the above four treatments. But, clearly a design containing (00), (10), and (11) is unbiased and this has fewer treatments. The reason is that:

$$L = \begin{bmatrix} \ell_1' \\ \ell_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_1(\rho) \\ R_2(\rho) \\ R_3(\rho) \\ R_4(\rho) \end{bmatrix} .$$

Here $\Gamma_1 = \{(00), (11)\}$, $\Gamma_2 = \{(10)\}$ and $\Gamma_1 \cup \Gamma_2 = \{(00), (10), (11)\}$, resulting in the FFA$(2,2;\ 3;\ 3;\ r_{00} = 1,\ r_{10} = 1,\ r_{11} = 1)$. Any design containing $\Gamma_1 \cup \Gamma_2$ is an unbiased design.

It is clear that the general problem of characterizing unbiased designs in a useful way is not solved. We will next give results in some special cases. Before doing this we need the following definition.

Definition 4.1: An unbiased design for $L\beta_\rho$ is said to be minimal if the number of treatments in the design is minimal. Such a design will be referred to as a minimal unbiased design.

5. Characterization of Minimal Unbiased Designs for $L\beta_\rho$ With No Assumptions On β_ρ

Let L be an $s \times N$ matrix and suppose that the experimenter is interested in estimating $L\beta_\rho$. The following algorithm generates a unique minimal unbiased design for $L\beta_\rho$. Since L is in the row space of X_ρ, we may write

$$L' = X_\rho' C \tag{5.1}$$

where C is an $N \times s$ matrix of coefficients. Since X_ρ is orthogonal, the unique solution for C is

$$C = X_\rho L' \ . \tag{5.2}$$

Hence, the unique minimal design is given by those treatments i for which the i^{th} row of C is not all zeros. Clearly any design containing this minimal design will also be an unbiased design.

Example 5.1. Consider the 3×3 factorial such that the coded levels of the factors are $\{0,1,2\}$. Then under the orthogonal polynomial model the design matrix X_ρ is equal to

$$X_\rho = \begin{bmatrix} \frac{1}{3} & \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{2} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-\sqrt{2}}{3} & \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{-1}{2} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{1}{6} \\ \frac{1}{3} & \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & 0 & \frac{-\sqrt{2}}{3} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{-1}{3} \\ \frac{1}{3} & 0 & \frac{-\sqrt{2}}{3} & 0 & \frac{-\sqrt{2}}{3} & 0 & 0 & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & 0 & \frac{-\sqrt{2}}{3} & 0 & \frac{-1}{\sqrt{3}} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{-1}{2} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-\sqrt{2}}{3} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{3}} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{6} \end{bmatrix}$$

Let $L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$; then from the above it follows that the minimal design for $L\beta_\rho$ is determined by the nonzero rows of C which are shown below:

$$C = X_\rho L' = \begin{bmatrix} \frac{-1}{\sqrt{6}} & \frac{1}{2} & \frac{-1}{2\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{2} & \frac{-1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{1}{2\sqrt{3}} \end{bmatrix} .$$

Hence the unique minimal design is

$$\Gamma = \{(00),\ (10),\ (20),\ (02),\ (12),\ (22)\} .$$

Example 5.2. Consider the 3×3 factorial such that the coded levels of the factors are $\{0,1,3\}$ and $\{0,2,3\}$ respectively. Then under the orthogonal polynomial model the design matrix X_ρ is equal to

$$\begin{bmatrix}
\frac{1}{3} & \frac{-4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{20}{42} & \frac{-4}{\sqrt{588}} & \frac{-10}{\sqrt{588}} & \frac{2}{14} \\[6pt]
\frac{1}{3} & \frac{-1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{5}{42} & \frac{-1}{\sqrt{588}} & \frac{15}{\sqrt{588}} & \frac{-3}{14} \\[6pt]
\frac{1}{3} & \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{-25}{42} & \frac{5}{\sqrt{588}} & \frac{-5}{\sqrt{588}} & \frac{1}{14} \\[6pt]
\frac{1}{3} & \frac{-4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{-4}{42} & \frac{12}{\sqrt{588}} & \frac{2}{\sqrt{588}} & \frac{-6}{14} \\[6pt]
\frac{1}{3} & \frac{-1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{-1}{42} & \frac{3}{\sqrt{588}} & \frac{-3}{\sqrt{588}} & \frac{9}{14} \\[6pt]
\frac{1}{3} & \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{5}{42} & \frac{-15}{\sqrt{588}} & \frac{1}{\sqrt{588}} & \frac{-3}{14} \\[6pt]
\frac{1}{3} & \frac{-4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{-16}{42} & \frac{-8}{\sqrt{588}} & \frac{8}{\sqrt{588}} & \frac{4}{14} \\[6pt]
\frac{1}{3} & \frac{-1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{-4}{42} & \frac{-2}{\sqrt{588}} & \frac{-12}{\sqrt{588}} & \frac{-6}{14} \\[6pt]
\frac{1}{3} & \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{20}{42} & \frac{10}{\sqrt{588}} & \frac{4}{\sqrt{588}} & \frac{2}{14}
\end{bmatrix}$$

Let $L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ as in example 5.1; then from the above

it follows that the minimal design for $L\beta_\rho$ is determined by the nonzero rows of C which are shown below.

$$C = X_\rho L' = \begin{bmatrix} \frac{-5}{\sqrt{42}} & \frac{20}{42} & \frac{-10}{\sqrt{588}} \\ \frac{-5}{\sqrt{42}} & \frac{5}{42} & \frac{15}{\sqrt{588}} \\ \frac{-5}{\sqrt{42}} & \frac{-25}{42} & \frac{-5}{\sqrt{588}} \\ \frac{1}{\sqrt{42}} & \frac{-4}{42} & \frac{2}{\sqrt{588}} \\ \frac{1}{\sqrt{42}} & \frac{-1}{42} & \frac{-3}{\sqrt{588}} \\ \frac{1}{\sqrt{42}} & \frac{5}{42} & \frac{1}{\sqrt{588}} \\ \frac{4}{\sqrt{42}} & \frac{-16}{42} & \frac{8}{\sqrt{588}} \\ \frac{4}{\sqrt{42}} & \frac{-4}{42} & \frac{-12}{\sqrt{588}} \\ \frac{4}{\sqrt{42}} & \frac{20}{42} & \frac{4}{\sqrt{588}} \end{bmatrix}$$

Hence, the unique minimal design is the MFA which consists of the entire set of nine combinations $\{(00), (02), (03), (10), (12), (13), (30), (32), (33)\}$. It should be noted that the equal spacing of levels in example 5.1 allowed the use of 6 treatments to estimate $L\beta_\rho$ whereas the unequal spacing in this example required that all 9 observations be utilized. Other contrasts may result in a different minimal design.

6. Characterization of Minimal Unbiased Designs For $L\beta_1$ Under The Assumption That $\beta_2 = 0$

We assume that $\beta_\rho' = (\beta_1' \vdots \beta_2' = 0)$ where β_1 is a $p \times 1$ vector of parameters.

The problem here is to find a minimal unbiased design for $L_1\beta_1$. Recall that the model in this case is equal to

(6.1) $$E\left[Y_\rho\right] = X_{\rho 1}\beta_1 \ .$$

For L_1 to be in the row space of $X_{\rho 1}$ there must exist a matrix C_1 such that

(6.2) $$L_1' = X_{\rho 1}' C_1 \ .$$

Clearly a solution for C_1 is given by

(6.3) $$C_1 = X_{\rho 1} L_1', \text{ since } X_{\rho 1}' X_{\rho 1} = I \ .$$

Hence an unbiased design for $L_1\beta_1$ is given by those treatments i for which the i^{th} rows of C_1 are nonzero rows. Such a design is not necessarily minimal as the following example indicates.

Example 6.1. Consider the 2 x 2 factorial with coded levels $\{0,1\}$, and let $\beta_1' = \left(\phi_1^0\phi_2^0,\ \phi_1^1\phi_2^0\right)$. The induced model is then given by

$$E\begin{bmatrix} y_{00} \\ y_{10} \\ y_{01} \\ y_{11} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \phi_1^0\phi_2^0 \\ \phi_1^1\phi_2^0 \end{bmatrix} = X_{\rho 1}\beta_1 \ .$$

If $L_1 = \frac{1}{2}(1\ -1)$ then a solution for C_1 is given by

$$C_1 = X_{\rho 1} L_1' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

Thus an unbiased design determined by C_1 is $\{(00),\ (01)\}$. But clearly the designs $\{(00)\}$ and $\{(01)\}$ are two minimal unbiased designs for this problem. This clearly follows from the non-uniqueness of C_1 which in turn reflects the dependency of the rows of $X_{\rho 1}$. For these minimal designs the solutions for the coefficient matrices are

$$C_1^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_1^{**} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

From the above it follows that the problem of determining unbiased designs in this setting is solved by finding those solutions to C_1 in the equation $X_{\rho 1}' C_1 = L_1'$ for which C_1 has maximum number of rows with all elements equal to zero. In the literature this problem is known as the <u>non-singularity problem</u> in fractional replication when $L_1 = I_p$. As of the present, all these problems are unresolved.

<u>Remark</u>. So far, we have ignored the problem of estimating σ^2 if it is unknown. However, if the experimenter is interested in estimating this parameter, then the design should take repetitions and/or the addition of treatments into account.